

Geometric Product Formula for Charged Accelerating Black Hole

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Abstract

We evaluate the geometric product formula i.e. area (or entropy) product formula of outer horizon (\mathcal{H}^+) and inner horizon (\mathcal{H}^-) for charged accelerating black hole. We find that mass-independent area functional relation of \mathcal{H}^\pm for this black hole in terms of black hole charge, acceleration, cosmological constant and *cosmic string tension* respectively. We also compute the *Penrose inequality* for this black hole. Finally we compute the specific heat for this BH to determine the local thermodynamic stability of this black hole. Under certain criterion the black hole displayed second order phase transition.

1 Introduction

Black Hole (BH) is an universal thermal object [1, 2]. Their thermodynamic property is also universal. New thermodynamic product relations of event horizon area and Cauchy horizon (\mathcal{H}^-) area for several BHs have been found are also universal. Why this relation is called universal because the product of area of multihorizons particularly two physical horizons namely \mathcal{H}^\pm is mass-independent. This is why this is an interesting topic in recent years in the scientific community particularly in the GR community [3] and in the string theory community [4].

In case of Kerr-Newman (KN) BH [3] which is a electrovacuum solution of Einstein's equations, it has been shown that the product of inner and outer horizon area of \mathcal{H}^\mp should read

$$\mathcal{A}_- \mathcal{A}_+ = 64\pi^2 J^2 + 16\pi^2 Q^4 . \quad (1)$$

where \mathcal{A}_- and \mathcal{A}_+ are area of Cauchy horizon and event horizon. This relation indicates that the universal product depends only quantized angular momentum and quantized charges respectively.

When charge vanishes that means for Kerr BH, this relation is reduced to more simpler form as

$$\mathcal{A}_- \mathcal{A}_+ = 64\pi^2 J^2 . \quad (2)$$

and it indicates that the universal product depends only quantized angular momentum parameter.

Again when angular momentum vanishes that means for spherically symmetric charged BH, the above relation should read

$$\mathcal{A}_- \mathcal{A}_+ = 16\pi^2 Q^4 . \quad (3)$$

and it implies that the universal product depends only quantized charge parameter.

In some cases this universal product relationship simply fails. In this cases more complicated function of \mathcal{H}^\pm area is mass-independent but this is not guaranteed that they are universal but we can hope that this thermodynamic product relation may turn out to be universal. For instance, Visser [5] showed that after introducing the cosmological constant the product of inner and outer horizon area for Schwarzschild-de Sitter (Kottler) BH is not mass-independent. Similarly, in case of charged-AdS BH the product of \mathcal{H}^\pm area (computed perturbatively) is not mass-independent. Thus Visser [5] concluded that mass-independence of product of \mathcal{H}^\pm area is not generic.

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Since one can not derive simple area product for AdS BH spacetime as in KN, Kerr and charged BH therefore one could derive more complicated function of two physical horizons area that might be interesting as well as one finds it is mass-independent then these relations could turn to be *universal*.

For example in case of charged-dS BH [5], the mass-independent relation becomes

$$\frac{(\sqrt{\mathcal{A}_+} + \sqrt{\mathcal{A}_-} + \sqrt{\mathcal{A}_{--}})\sqrt{\mathcal{A}_+}\sqrt{\mathcal{A}_-}\sqrt{\mathcal{A}_{--}}}{\mathcal{A}_+ + \mathcal{A}_- + \mathcal{A}_{--} + \sqrt{\mathcal{A}_+\mathcal{A}_-} + \sqrt{\mathcal{A}_-\mathcal{A}_{--}} + \sqrt{\mathcal{A}_{--}\mathcal{A}_+}} = 4\pi Q^2. \quad (4)$$

where \mathcal{A}_{--} denotes cosmological horizon.

Similarly, for RN-AdS BH, the mass-independent relation should read [5]:

$$\left[1 + \frac{|\Lambda|}{12\pi} (\mathcal{A}_+ + \mathcal{A}_- + \sqrt{\mathcal{A}_+\mathcal{A}_-})\right] \sqrt{\mathcal{A}_+\mathcal{A}_-} = 4\pi Q^2. \quad (5)$$

Finally, for KN-AdS BH the mass-independent relation becomes [6]

$$\begin{aligned} & \left[1 + \frac{|\Lambda|}{6\pi}(\mathcal{A}_+ + \mathcal{A}_- + 4\pi Q^2) + \frac{|\Lambda|^2}{144\pi^2}(\mathcal{A}_+^2 + \mathcal{A}_+\mathcal{A}_- + \mathcal{A}_-^2)\right] \mathcal{A}_+\mathcal{A}_- \\ & = 64\pi^2 J^2 + 16\pi^2 Q^4. \end{aligned} \quad (6)$$

These three relations suggest that they are mass-independent as well as they are universal relation which relates quantized charge, quantized angular momentum, cosmological constant and area of \mathcal{H}^\pm .

Note that all the quasilocal quantities in the Eq. 1, Eq. 2 and Eq. 3 are evaluated locally on the two physical BH horizons. These relations also valid for perturbed BHs (i.e. BH with surrounding of ring of matter) as well, this is why they are more interesting. It may be noted that the *universal* term is used throughout the manuscript for relations that are mass-independent.

However in this work, we wish to extend our analysis for charged accelerating AdS BH following the previous work [5, 6, 7, 8, 9, 10]. By explicit and exact calculation, we show that there would be some more complicated combination of inner and outer horizons area that are mass-independent. But this is not straightforward as simple product of \mathcal{H}^\pm area as we have seen in case of Kerr, KN and charged BH. We also study other thermodynamic properties particularly the local thermodynamic stability by computing the specific heat. Under appropriate condition the BH possesses second order phase transition.

The interesting properties of the charged AdS BH is that it has accelerating horizon and the outer horizon possesses conical singularity [11]. In spite of that the BH is satisfied the usual first law of BH thermodynamics and the entropy is proportional to the usual area divided by four. But the cosmological horizon don't have accelerating horizon. This type of BH is said to be *slowly* accelerating BH. The other novel properties of this accelerating BH is that it is described by the C metric [12, 13, 15, 16]. Again the C metric has some peculiar features in the sense that it accelerates by pulling with a 'cosmic string' which described by a 'conical deficit in the spacetime' which connects the outer horizon of the BH to infinity [11]. Actually the idea of vacuum C metric was first given by Levi-Civita in 1918 [17]. Then it was rediscovered by Newman and Tamburino in 1961 [18], also by Robinson and Trautman [20] in same years, and by Ehlers and Kundt [19] in 1963 but there were no explanation given. First Robinson and Trautman discovered the interesting features of this metric is that it emits gravitational radiation. Later Podolsky et al. [21] studied the gravitational and electromagnetic radiation emitted by the uniformly accelerated charged BH in AdS spacetime.

2 Thermodynamic properties of Charged Accelerating BH:

The metric of the charged accelerating BH [16, 11] is described by

$$ds^2 = \frac{1}{\Omega^2} \left[-\mathcal{F}(r)dt^2 + \frac{dr^2}{\mathcal{F}(r)} + r^2 \left(\frac{d\theta^2}{\mathcal{G}(\theta)} + \mathcal{G}(\theta)\sin^2\theta \frac{d\phi^2}{K^2} \right) \right]. \quad (7)$$

with the gauge potential $F = dB$ and $B = -\frac{q}{r}dt$, and where

$$\mathcal{F}(r) = (1 - \chi^2 r^2) \left(1 - \frac{2m}{r} + \frac{q^2}{r^2} \right) + \frac{r^2}{\ell^2}, \quad (8)$$

$$\mathcal{G}(\theta) = 1 + 2m\chi \cos\theta + q^2 \chi^2 \cos^2\theta. \quad (9)$$

and the conformal factor is $\Omega = 1 + \chi r \cos\theta$. It determines the conformal infinity of the AdS BH. The quantities m and q are BH mass and BH charge respectively, $\chi > 0$ determines the acceleration of the BH and $-\frac{\Lambda}{3} = \frac{1}{\ell^2}$ where ℓ is the radius of the AdS BH. It should be noted that when $\chi < \frac{1}{\ell}$, a single BH is present with single horizon [14] whereas when $\chi > \frac{1}{\ell}$, two BHs of opposite charge are separated by accelerating horizon [15, 22] and when $\chi = \frac{1}{\ell}$ is a special case and it has been explicitly described in [23]. To obtain the angular coordinates as usual form on \mathcal{S}^2 we have set the restriction $m\chi < \frac{1}{2}$.

Now we discuss the angular part of the metric and the properties of $\mathcal{G}(\theta)$ at the North pole ($\theta = 0$) and South pole ($\theta = \pi$). In general, $K = \mathcal{G}(\theta)$ but at North pole fixed with $K = K_N = 1 + 2m\chi + q^2\chi^2$ and at South pole $K_S = 1 - 2m\chi + q^2\chi^2$ that indicates the metric is regular at North pole and South pole but there is a ‘conical deficit’ which is $\delta = \frac{8\pi m\chi}{1 + 2m\chi + q^2\chi^2}$ and which corresponds to a ‘cosmic string’ [11] with tension $\mu = \frac{\delta}{8\pi} = \frac{m\chi}{1 + 2m\chi + q^2\chi^2}$.

Thus the C metric is described by the five physical parameters: the mass m , the charge q , the negative cosmological constant $-\Lambda = \frac{3}{\ell^2}$, the acceleration χ and the ‘tension of the cosmic string’ on each axis which is represented by the periodicity of the angular coordinate. More discussion about the thermodynamic properties (particularly first law of BH thermodynamics, Smarr mass formula, thermodynamic volume, Gibbs free energy and Reverse isoperimetric inequality) of the C metric could be found details in [11].

Now we evaluate the radii of BH horizons by imposing the condition $\mathcal{F}(r_i) = 0$ i.e.

$$(1 - \chi^2 \ell^2) r_i^4 + 2m\ell^2 \chi^2 r_i^3 + \ell^2 (1 - \chi^2 q^2) r_i^2 - 2m\ell^2 r_i + q^2 \ell^2 = 0. \quad (10)$$

Applying the Vieta theorem we find

$$\sum_{i=1}^4 r_i = -\frac{2m\ell^2 \chi^2}{(1 - \chi^2 \ell^2)}. \quad (11)$$

$$\sum_{1 \leq i < j \leq 4} r_i r_j = \ell^2 \frac{(1 - \chi^2 q^2)}{(1 - \chi^2 \ell^2)}. \quad (12)$$

$$\sum_{1 \leq i < j < k \leq 4} r_i r_j r_k = \frac{2m\ell^2}{(1 - \chi^2 \ell^2)}. \quad (13)$$

$$\sum_{1 \leq i < j < k < l \leq 4} r_i r_j r_k r_l = \frac{q^2 \ell^2}{(1 - \chi^2 \ell^2)}. \quad (14)$$

Eliminating the mass parameter we obtain single mass-independent relation as

$$r_1 r_2 + \frac{q^2 \ell^2}{(1 - \chi^2 \ell^2) r_1 r_2} + (r_1 + r_2)^2 \times \left[r_1 r_2 \frac{\chi^2}{(1 + \chi^2 r_1 r_2)} - \frac{q^2 \ell^2 \chi^2}{r_1 r_2 (1 - \chi^2 \ell^2) (1 + \chi^2 r_1 r_2)} - 1 \right] = \ell^2 \frac{(1 - \chi^2 q^2)}{(1 - \chi^2 \ell^2)}. \quad (15)$$

In terms of two BH physical horizons area $\mathcal{A}_i = \frac{4\pi r_i^2}{K(1 - \chi^2 r_i^2)}$ (where $i = 1$ for event horizon and $i = 2$ for Cauchy horizon), the mass independent functional relationship is given by

$$\sqrt{\frac{K\mathcal{A}_1}{4\pi + K\chi^2\mathcal{A}_1}} \sqrt{\frac{K\mathcal{A}_2}{4\pi + K\chi^2\mathcal{A}_2}} + \frac{q^2 \ell^2}{(1 - \chi^2 \ell^2)} \sqrt{\frac{4\pi + K\chi^2\mathcal{A}_1}{K\mathcal{A}_1}} \sqrt{\frac{4\pi + K\chi^2\mathcal{A}_2}{K\mathcal{A}_2}} +$$

$$\begin{aligned}
& \left(\sqrt{\frac{K\mathcal{A}_1}{4\pi + K\chi^2\mathcal{A}_1}} + \sqrt{\frac{K\mathcal{A}_2}{4\pi + K\chi^2\mathcal{A}_2}} \right)^2 \times \\
& \left[\sqrt{\frac{K\mathcal{A}_1}{4\pi + K\chi^2\mathcal{A}_1}} \sqrt{\frac{K\mathcal{A}_2}{4\pi + K\chi^2\mathcal{A}_2}} \frac{\chi^2}{\left(1 + \chi^2 \sqrt{\frac{4\pi + K\chi^2\mathcal{A}_1}{K\mathcal{A}_1}} \sqrt{\frac{4\pi + K\chi^2\mathcal{A}_2}{K\mathcal{A}_2}}\right)} - 1 \right] - \\
& \left(\sqrt{\frac{K\mathcal{A}_1}{4\pi + K\chi^2\mathcal{A}_1}} + \sqrt{\frac{K\mathcal{A}_2}{4\pi + K\chi^2\mathcal{A}_2}} \right)^2 \times \\
& \sqrt{\frac{4\pi + K\chi^2\mathcal{A}_1}{K\mathcal{A}_1}} \sqrt{\frac{4\pi + K\chi^2\mathcal{A}_2}{K\mathcal{A}_2}} \left[\frac{q^2 \ell^2 \chi^2}{(1 - \chi^2 \ell^2) \left(1 + \chi^2 \sqrt{\frac{4\pi + K\chi^2\mathcal{A}_1}{K\mathcal{A}_1}} \sqrt{\frac{4\pi + K\chi^2\mathcal{A}_2}{K\mathcal{A}_2}}\right)} \right] = \ell^2 \frac{(1 - \chi^2 q^2)}{(1 - \chi^2 \ell^2)}
\end{aligned} \tag{16}$$

where,

$$K = 1 + 2m\chi + q^2\chi^2. \tag{17}$$

Eq. 15 is indeed mass independent but the difficulties arise when we write the expression in terms of area of the BH physical horizons because there is a factor K where a mass term is appeared. This mass term can be eliminated with the help of external condition i.e. *cosmic string tension* μ then the term K becomes

$$K = \left(\frac{1 + \mu}{1 - \mu} \right) (1 + q^2\chi^2). \tag{18}$$

Now the Eq. 16 is indeed mass independent. Now we may claim that this equation could turn out to be an universal quantity.

The BH entropy [11] is given by

$$\mathcal{S}_i = \frac{\mathcal{A}_i}{4} = \frac{\pi r_i^2}{K(1 - \chi^2 r_i^2)}. \tag{19}$$

and the electric potential on the horizon should read

$$\Phi_i = \frac{q}{r_i}. \tag{20}$$

Now the BH temperature¹ is given by

$$T_i = \frac{\mathcal{F}'(r)}{4\pi} = \frac{1}{4\pi} \left[\frac{2m}{r_i^2} - \frac{2q^2}{r_i^3} + 2m\chi^2 - 2\chi^2 r_i + \frac{2r_i}{\ell^2} \right]. \tag{21}$$

We know the famous Penrose inequality for Schwarzschild BH is given by

$$m \geq \sqrt{\frac{\mathcal{A}}{16\pi}}. \tag{22}$$

¹It seems to be a term missing in Eq. (9) in [11]. The term $T = \frac{m}{2\pi r_+^2} - \frac{e^2}{2\pi r_+^3} + \frac{mA^2}{2\pi} + \frac{r_+}{2\pi\ell^2}$ should read there $T = \frac{m}{2\pi r_+^2} - \frac{e^2}{2\pi r_+^3} + \frac{mA^2}{2\pi} - \frac{A^2 r_+}{2\pi} + \frac{r_+}{2\pi\ell^2}$.

This inequality for accelerating BH becomes

$$m \geq \frac{1}{2} \sqrt{\frac{K\mathcal{A}_i}{4\pi + K\chi^2\mathcal{A}_i}} \left[1 + q^2 \left(\frac{4\pi + K\chi^2\mathcal{A}_i}{K\mathcal{A}_i} \right) + \frac{K\mathcal{A}_i}{4\pi\ell^2} \right]. \quad (23)$$

This idea had first given in 1973 by Penrose [24] which is an important topic in GR which relates ADM mass (i.e. total mass of the spacetime) and the area of the event horizon. It is called Penrose inequality.

Local thermodynamic stability and phase transition particularly second order phase transition are important phenomena in BH thermodynamics and it can be determined by computing the specific heat which is calculated to be for accelerating BH:

$$C_i = 2\pi r_i^2 \frac{\left[1 - \frac{q^2}{r_i^2} + \frac{r_i^2}{\ell^2} \frac{(3 - \chi^2 r_i^2)}{(1 - \chi^2 r_i^2)^2} \right]}{\left[\frac{4\chi^2 r_i^4}{\ell^2 (1 - \chi^2 r_i^2)^2} + \frac{r_i^2}{\ell^2} \left(\frac{3 - \chi^2 r_i^2}{1 - \chi^2 r_i^2} \right) + \frac{3q^2}{r_i^2} - (1 + q^2\chi^2 + \chi^2 r_i^2) \right]}. \quad (24)$$

Thermodynamic stability requires that $C_i > 0$ and the criterion for specific heat diverges where second order phase transition occurs at

$$\frac{\left[\frac{4\chi^2 r_i^4}{\ell^2 (1 - \chi^2 r_i^2)^2} + \frac{r_i^2}{\ell^2} \left(\frac{3 - \chi^2 r_i^2}{1 - \chi^2 r_i^2} \right) + \frac{3q^2}{r_i^2} \right]}{(1 + q^2\chi^2 + \chi^2 r_i^2)} = 1. \quad (25)$$

3 Conclusion:

We have studied the thermodynamic properties of slowly accelerating BH consists of five parameters, the mass, the charge, the acceleration, the cosmological constant and cosmic string tension respectively. We have derived the mass-independent area (or entropy) functional relation of \mathcal{H}^\pm in terms of BH charge, acceleration, cosmological constant and cosmic string tension respectively. We hope that this complicated area functional relation of \mathcal{H}^\pm could turn out to be universal. We also derived the famous Penrose inequality for this slowly accelerating BH. Finally, we evaluated the criterion under which the BH showed the second order phase transition.

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